The p-Laplacian on Graphs with Applications in Image Processing and Classification

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1. Motivation

2. Introduction to weighted graphs

3. The graph p-Laplacian operator

4. Solving partial difference equations on graphs
1. Motivation

2. Introduction to weighted graphs

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4. Solving partial difference equations on graphs
(Nonlocal) image processing

Aim: Use nonlocal information for image processing

e.g., *denoising, inpainting, segmentation, …*
Surface processing

Aim: Approximate surfaces by meshes and process mesh data
e.g., in computer graphics, finite element methods, …

Image courtesy: Gabriel Peyré via http://www.cmap.polytechnique.fr/~peyre/geodesic_computations/
Surface processing
Aim: Approximate surfaces by discrete points and process point cloud data e.g., in 3D vision, augmented reality, surface reconstruction, …
Aim: Investigate interaction and processes in networks of arbitrary topology
e.g., in social networks, computer networks, transportation, …
Graph based methods

Goal:
Use graphs to perform

\[ \text{filtering, segmentation, inpainting, classification, \ldots} \]

on

\[ \text{data of arbitrary topology.} \]

Question:
How to translate \text{PDEs} and \text{variational methods} to graphs?
Related works


1. Motivation

2. Introduction to weighted graphs

3. The graph p-Laplacian operator

4. Solving partial difference equations on graphs
Introducing weighted graphs

A graph $G = (V, E, w)$ consists of:

- a finite set of *vertices* $V = (v_1, \ldots, v_n)$
- a finite set of *edges* $E \subseteq V \times V$
- a *weight function* $w : E \rightarrow [0, 1]$

We consider mainly *undirected*, weighted graphs in the following!
Vertex functions

A vertex function $f : V \rightarrow \mathbb{R}^N$ assigns each $v \in V$ a vector of features:

- grayscale value, RGB color vector
- 3D coordinates
- label

The space of vertex functions $H(V)$ is a Hilbert space with the norm:

$$\|f\| = \sqrt{\sum_{v_i \in V} \langle f(v_i), f(v_i) \rangle}$$
Weight functions

A weight function \( w : E \rightarrow [0, 1] \) assigns each \( e \in E \) a weight based on the similarity of respective node features.

To compute a weight \( w(u,v) = w(v,u) \) for nodes \( u,v \in V \) we need:

1. Symmetric distance function \( d(f(u),f(v)) = d(f(v),f(u)) \in \mathbb{R} \)
   - e.g., constant distance, \( l^p \) norms, patch distance, ...

2. Normalized similarity function \( s(d(f(u),f(v))) \in [0, 1] \)
   - e.g, constant similarity, probability density function, ...

Example:

\[
    w(u,v) = e^{-\frac{|d(u,v)|^2}{\sigma^2}}, \quad \text{with} \quad d(u,v) = \|p_k(u) - p_k(v)\|_2
\]
Patch distance

- Pixel of interest
- Intensity-based
- Patch-based
Graph construction methods

1. $\epsilon$-ball graph
Graph construction methods

2. k-Nearest Neighbour graph (directed):
Graph construction methods

2. $k$-Nearest Neighbour graph (directed):

(a) 

(b) $k=3$ 

(c) $k=15$
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Weighted finite differences on graphs

Let \((V, E, w)\) be a weighted graph and let \(f : V \rightarrow \mathbb{R}^N\) be a vertex function. The weighted (nonlocal) finite difference \(d_w : H(V) \rightarrow H(E)\) of \(f \in H(V)\) along an edge \((u,v) \in E\) is given as:

\[
d_w f(u, v) = \sqrt{w(u, v)}(f(v) - f(u))
\]

Then the weighted gradient of \(f\) in a vertex \(u\) is given as:

\[
(\nabla_w f)(u) = (\partial_v f(u))_{v \in V}, \quad \partial_v f(u) = \sqrt{w(u, v)}(f(v) - f(u))
\]

Similarly, we can introduce the weighted upwind gradient as:

\[
\partial_v^\pm f(u) = \sqrt{w(u, v)}(f(v) - f(u))^\pm
\]

\[
(\nabla_w^\pm f)(u) = \left(\partial_v^\pm f(u)\right)_{v \in V}
\]

using the notation \(x^+ = \max(0,x)\) and \(y^- = -\min(0,x)\).
Special case: grayscale image

Let $G = (V, E, w)$ be a directed 2-neighbour grid graph with the weight function $w$ chosen as:

$$\partial_v f(u) = \sqrt{w(u, v)} (f(v) - f(u)) \quad w(u, v) = \begin{cases} \frac{1}{h^2}, & \text{if } u \sim v \\ 0, & \text{else} \end{cases}$$

Weighted finite differences correspond to forward differences!
Adjoint operator and divergence

Let $f \in H(V)$ be a vertex function and let $G \in H(E)$ be an edge function. One can deduce the adjoint operator $d^*_w : H(E) \rightarrow H(V)$ of $d_w : H(V) \rightarrow H(E)$ by the following property:

$$\langle d_w f, G \rangle_{H(E)} = \langle f, d^*_w G \rangle_{H(V)}$$

Then the divergence $\text{div}_w : H(E) \rightarrow H(V)$ of $G$ in a vertex $u$ is given as:

$$\text{div}_w G(u) = -d^*_w G(u) = \sum_{v \sim u} \sqrt{w(u,v)}(G(u,v) - G(v,u))$$

We have in particular the following conservation law:

$$\sum_{u \in V} \text{div}_w G(u) = 0$$
PdE framework for graphs

We further want to translate PDEs to graphs and formulate them as partial difference equations (PdEs).

Example: Mimic heat equation on graphs

\[
\frac{\partial f}{\partial t}(u, t) = \Delta_w f(u, t) \quad \text{with} \quad \Delta_w f(u) = \sum_{v \sim u} w(u, v)(f(v) - f(u))
\]

\[f(u, t = 0) = f_0(u)\]
The variational p-Laplacian

The variational p-Laplacian is a quasilinear elliptic partial differential operator of second order and can be given for \( u \in C^2(\Omega) \) as:

\[
\Delta_p u = \text{div} \left( \frac{\nabla u}{|\nabla u|^{2-p}} \right), \quad 1 \leq p < \infty
\]

Note that for \( p < 2 \) the p-Laplacian has critical points in \( \nabla u = 0 \).

The p-Laplacian equation with Dirichlet boundary conditions \( \Delta_p u = 0 \) can be derived as Euler-Lagrange equation of the minimization problem:

\[
E(u) = \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p \, dx
\]

Applications:

- **Modeling:** Biological/physical processes with diffusion, e.g., heat equation
- **Image processing:** total variation (\( p=1 \)) and Tikhonov regularization (\( p=2 \))
The graph p-Laplacian

Let \( f \in H(V) \) be a vertex function. Then the isotropic graph p-Laplacian operator in an vertex \( u \) is given as:

\[
(\Delta^i_{w,p} f)(u) = \frac{1}{2} \text{div}_w \left( ||\nabla_w f||^{p-2} d_w f \right)(u)
\]

We can also define the anisotropic graph p-Laplacian operator in an vertex \( u \) as:

\[
(\Delta^a_{w,p} f)(u) = \frac{1}{2} \text{div}_w \left( |d_w f|^{p-2} d_w f \right)(u) = \sum_{v \sim u} (w(u,v))^{p/2} |f(v) - f(u)|^{p-2}(f(v) - f(u))
\]
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Applications

Diffusion processes on graphs

One important class of PdEs on graphs are diffusion processes of the form:

\[
\begin{align*}
\frac{\partial f(u,t)}{\partial t} &= \Delta_{w,p}^a f(u, t), \\
f(u, t = 0) &= f_0(u),
\end{align*}
\]

Applying forward Euler time discretization leads to an iterative scheme:

\[
f^{n+1}(u) = f^n(u) + \Delta t \sum_{v \sim u}(w(u, v))^{p/2}|f(v) - f(u)|^{p-2}(f(v) - f(u))
\]

Maximum norm stability can be guaranteed under the CFL condition:

\[
1 \geq \Delta t \sum_{v \sim u}(w(u, v))^{p/2}|f(v) - f(u)|^{p-2}
\]
Denoising

Local

Local + weight
Denoising

Noisy data

Total variation denoising (local)
600 iterations

Total variation denoising (nonlocal)
1200 iterations
Interpolation problems on graphs

Another class of PdEs on graphs are interpolation problems of the form:

\[
\begin{cases}
\Delta^a_{w,p} f(u) = 0, & \text{for } u \in A, \\
f(u) = g(u), & \text{for } u \in \partial A.
\end{cases}
\]

for which \( A \subseteq V \) is a subset of vertices and \( \partial A = V \setminus A \) and the given information \( g \) are application dependent.

Solving this Dirichlet problem amounts in finding the stationary solution of a diffusion process with fixed boundary conditions.

\[
\begin{cases}
\frac{\partial f(u,t)}{\partial t} = \Delta^a_{w,p} f(u,t), & \text{for } u \in A \\
f(u) = g(u), & \text{for } u \in \partial A.
\end{cases}
\]
Interactive segmentation
Interactive segmentation
Inpainting

Original image

Inpainting region

Local inpainting

Nonlocal inpainting
Inpainting

Original image
Inpainting region
Local inpainting
Nonlocal inpainting
Semi supervised classification
Semi supervised classification
Summary

1) Graph framework unifies local and nonlocal methods

2) PdEs / discrete variational models applicable to data of arbitrary topology

3) Experimental results were demonstrated for:
   - Denoising
   - Inpainting
   - Semi supervised segmentation
   - Classification

Thank you for your attention! Any questions?
Discrete optimization problems on graphs

We want to mimic variational models on graphs and formulate them as discrete optimization problems.

**Example:** ROF TV denoising model

\[
\|u\|_{TV,p} = \sum_{x_i \in \mathcal{V}} \| (\nabla_w u)(x_i) \|_p = \sum_{x_i \in \mathcal{V}} \left( \sum_{x_j \sim x_i} |(d_w u)(x_i, x_j)|^p \right)^{1/p}, \quad 1 \leq p < \infty
\]

\[
\|u\|_{TV,\infty} = \sum_{x_i \in \mathcal{V}} \| (\nabla_w u)(x_i) \|_\infty = \sum_{x_i \in \mathcal{V}} \sup_{x_j \sim x_i} |(d_w u)(x_i, x_j)|, \quad p = \infty
\]

Find a minimizer \( u \in \mathcal{H}(\mathcal{V}) \) of the energy

\[
E : \mathcal{H}(\mathcal{V}) \to \mathbb{R}, \quad E(u) = \lambda \| u - f \|_2^2 + \| u \|_{TV}
\]

→ Unified formulation for both local and nonlocal problems.

Rudin, Osher, Fatemi:
Variational $p$-Laplacian and $\infty$-Laplacian

The variational $p$- and $\infty$-Laplacians are quasilinear elliptic partial differential operators of second order and can be given for $u \in C^2(\Omega)$ as:

$$\Delta_p u = \text{div} \left( \frac{\nabla u}{|\nabla u|^{2-p}} \right), \quad 1 \leq p < \infty$$

$$\Delta_{\infty} u = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

Note that for $p < 2$ the $p$-Laplacian has critical points in $\nabla u = 0$.

The $p$-Laplacian equation with Dirichlet boundary conditions $\Delta_p u = 0$ can be derived as Euler-Lagrange equation of the minimization problem:

$$E(u) = \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p \, dx$$

Applications:

- Modeling: Biological/physical processes with diffusion, e.g., heat equation
- Image processing: total variation ($p=1$) and Tikhonov regularization ($p=2$)
Game p-Laplacian and $\infty$-Laplacian

The game p- and $\infty$-Laplacian are also known as normalized Laplacian and they are given as:

$$\Delta_p^G u = \frac{1}{p} |\nabla u|^{2-p} \Delta_p u = \frac{1}{p} |\nabla u|^{2-p} \text{div} \left( \frac{\nabla u}{|\nabla u|^{2-p}} \right), \quad 1 \leq p < \infty$$

$$\Delta_\infty^G u = |\nabla u|^{-2} \Delta_\infty u = |\nabla u|^{-2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

The game p-Laplacian equation $\Delta_p^G u = 0$ is singular whenever $p \neq 2$. One is able to recover the mean curvature flow (p=1) and 2-Laplacian (p=2):

$$\Delta_1^G u = \text{div} \left( \frac{\nabla u}{|\nabla u|} \right) |\nabla u| \quad \Delta_2^G u = \frac{1}{2} \Delta u$$

The game p-Laplacian can be expressed as convex combination:

$$\Delta_p^G u = \frac{1}{p} \Delta_1^G u + \frac{1}{q} \Delta_\infty^G u, \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad 1 < p, q < \infty$$

Applications:

- Game theory: Tug-of-War game with noise for $p=\infty$
Nonlocal $p$-Laplacian

The nonlocal $p$-Laplacian for a given continuous, normalized, and radial function $\mu: \mathbb{R}^n \to \mathbb{R}$ with compact support is given as:

$$\mathcal{L}_p u(x) = \int_{\Omega} \mu(x - y) |u(y) - u(x)|^{p-2}(u(y) - u(x)) \, dy, \quad 1 \leq p < \infty$$

If the convolution kernel $\mu$ is chosen as $\mu(x - y) = \frac{1}{|x - y|^{n+ps}}$, $p \geq 1$, $0 < s < 1$ one derives the fractional $p$-Laplacian:

$$\mathcal{L}_p u(x) = \int_{\Omega} \frac{|u(y) - u(x)|^{p-2}}{|x - y|^{n+ps}}(u(y) - u(x)) \, dy, \quad 1 \leq p < \infty$$

In the case $p = \infty$ this operator corresponds to the Hölder $\infty$-Laplacian:

$$\mathcal{L}_\infty u(x) = \max_{y \in \Omega, y \neq x} \left( \frac{u(y) - u(x)}{|y - x|^{\alpha}} \right) + \min_{y \in \Omega, y \neq x} \left( \frac{u(y) - u(x)}{|y - x|^{\alpha}} \right)$$

Applications:
- Image processing: Nonlocal regularization
- Modeling: Quantum phenomena in physics or population dynamics
A novel Laplacian operator on graphs

Let \( G = (V, E, w) \) be a weighted undirected graph and \( \alpha, \beta : \mathbb{H}(V) \rightarrow \mathbb{R}^N \) vertex functions with \( \alpha(u) + \beta(u) = 1 \) for all \( u \in V \). We propose a novel Laplacian operator on \( G \) as:

\[
\mathcal{L}_{w,p} f(u) = \begin{cases} 
\alpha(u) \| (\nabla^+_w f)(u) \|_{p-1}^{p-1} - \beta(u) \| (\nabla^-_w f)(u) \|_{p-1}^{p-1}, & 2 \leq p < \infty \\
\alpha(u) \| (\nabla^+_w f)(u) \|_{\infty}, & p = \infty 
\end{cases}
\]

From previous works it gets clear that:

\[
\Delta_{w,p} f(u) = \| \nabla^+_w f(u) \|_{p-1}^{p-1} - \| \nabla^-_w f(u) \|_{p-1}^{p-1}
\]

A simple factorization leads to this representation:

\[
\mathcal{L}_{w,p} f(u) = 2 \min(\alpha(u), \beta(u)) \Delta_{w,p} f(u) + (\alpha(u) - \beta(u))^+ \| (\nabla^+_w f)(u) \|_{p-1}^{p-1} - (\alpha(u) - \beta(u))^- \| (\nabla^-_w f)(u) \|_{p-1}^{p-1}, \quad 2 \leq p < \infty
\]

\[
\mathcal{L}_{w,\infty} f(u) = 2 \min(\alpha(u), \beta(u)) \Delta_{w,\infty} f(u) + (\alpha(u) - \beta(u))^+ \| (\nabla^+_w f)(u) \|_{\infty} - (\alpha(u) - \beta(u))^+ \| (\nabla^-_w f)(u) \|_{\infty}
\]

Observation:

Novel operator is a combination of \( p \)-Laplacian and upwind gradient operators on graphs.